Galilean equations for massless fields

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Abstract
Galilei-invariant equations for massless fields are obtained via contractions of
relativistic wave equations. It is shown that the collection of non-equivalent
Galilei-invariant wave equations for massless fields with spin equal to 1 and 0
is very rich and corresponds to various contractions of the representations of
the Lorentz group to those of the Galilei ones. It describes many physically
consistent systems, e.g., those of electromagnetic fields in various media or
Galilean Chern–Simons models. Finally, classification of all linear and a big
group of nonlinear Galilei-invariant equations for massless fields is presented.

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1. Introduction

It has already been observed by Le Bellac and Lévy-Leblond [1] in 1973 that a non-relativistic
limit of the Maxwell equations is not unique. According to them [1, p 218], the term 'non-
relativistic' means 'in agreement with the principle of Galilei relativity'. Moreover, they
claimed that there exist two Galilei invariant theories of electromagnetism which can be
obtained by appropriate limiting procedures starting with the Maxwell theory.

The words 'Galilean electromagnetism' themselves, introduced in [1], looked rather
strange since it is pretty well known that electromagnetic phenomena are in perfect accordance
with the Einstein relativity principle. On the other hand, physicists are always interested in
whether non-relativistic approximations are adequate, which makes the results of the paper
[1] quite popular. The importance of such results are emphasized by the fact that a correct
definition of a non-relativistic limit is by no means a simple problem, in general, and in the
case of massless fields in particular (see, for example, [2]).

Analyzing the contents of the main impact journals in theoretical and mathematical
physics, one finds that Galilean aspects of electrodynamics are evergreen subjects.
Various approaches to the Galilei-invariant theories were discussed briefly in [3]. Galilei
electromagnetism was discussed in papers [4–6] by using the *reduction approach* in which

the Maxwell equations were generalized into a (1+4)-dimensional Minkowski space and then reduced to the Galilei-invariant equations. Such reduction is based on the fact that the Galilei group is a subgroup of the generalized Poincaré group (i.e., of the group of motions of the flat (1+4)-dimensional Minkowski space). For reduction of representations of the group $P(1, 4)$ to those of the Galilei group see [7, 8].

In spite of that, Galilei electromagnetism contains constantly many unsolved problems, for example, the question of a complete description of all Galilean theories for vector and scalar massless fields. In fact, the solution of this problem is the main issue of the present paper.

In paper [3], the indecomposable representations of the homogeneous Galilei group HG(1, 3) were derived, namely, all those which when restricted to representations of the rotation subgroup of the group HG(1, 3), are decomposed to the spins 0 and 1 representations. Moreover, their connection with representations of the Lorentz group via the Inönü–Wigner contractions [9] were studied in [3, 10]. These results allow us to complete the classification of the wave equations describing both massive and massless fields.

In the present paper, we use our knowledge of indecomposable representations of the homogeneous Galilei algebra hg(1, 3) from [3, 10] to derive the Galilei invariant equations for vector and scalar massless fields. We shall show that, in contrast to the corresponding relativistic equations for which there are only two possibilities, namely, the Maxwell equations and the equations for the longitudinal massless field, the number of possible Galilei equations is huge. The principal description of such equations for vector and scalar fields is presented in the appendix where a complete list of relative differential invariants is presented. Among them there are equations for fields with more or less components than in the Maxwell equations.

These results can be clearly interpreted via representation and contraction theories. As was proved in the papers [3, 10], there is a large variety of possible contractions of diverse representations of the Lorentz group to those of the Galilei one and, consequently, many non-equivalent Galilei massless fields. In the following sections, we use these results from [3, 10] to describe connections of the relativistic and Galilei theories for massless fields. These connections appear to be rather non-trivial: some completely decoupled relativistic systems can be contracted to coupled Galilei ones.

We pay special attention to nonlinear Galilean systems for massless fields. In particular, Galilei-invariant Born–Infeld and Chern–Simons systems are deduced. A Galilei-invariant Lagrangian in (1+3)- dimensional space which includes a Chern–Simons term bilinear in field components is discussed. It is shown that, like in the initial Chern–Simons model [18], the Galilean Chern–Simons term does not affect the energy–momenta tensor.

In sections 2 and 3, we present some results of paper [3] concerning the classification of indecomposable representations of the homogeneous Galilei group and the contractions of related representations of the Lorentz group. These results are used in section 4 to classify all non-equivalent Galilei-invariant equations of first order for vector and scalar massless fields. In section 5, nonlinear Galilei-invariant equations are derived including Galilean Born–Infeld and Chern–Simons systems. Finally, the results of classification are summarized and discussed in section 6.

2. Indecomposable representations of the homogeneous Galilei group

The Galilei group G(1, 3) consists of the following transformations of temporal and spatial variables:

$$
\begin{aligned}
t &\rightarrow t' = t + a, \\
\mathbf{x} &\rightarrow \mathbf{x}' = \mathbf{R}\mathbf{x} + \mathbf{v}t + \mathbf{b},
\end{aligned}
\tag{1}
$$

where a, $\mathbf{b}$ and $\mathbf{v}$ are real parameters of time translation, space translations and pure Galilei transformations respectively and $\mathbf{R}$ is a rotation matrix.

The homogeneous Galilei group HG(1, 3) is a subgroup of the group G(1, 3) leaving invariant the point $\mathbf{x} = (0, 0, 0)$ at time $t = 0$. It is formed by space rotations and pure Galilei transformations, i.e., by transformations (1) with $a = 0$ and $\mathbf{b} = 0$.

The Lie algebra hg(1, 3) of the homogeneous Galilei group includes six basis elements, namely, three generators $S_a, a = 1, 2, 3$ of the rotation subgroup and three generators η_a of the Galilei boosts. These basis elements satisfy the following commutation relations:

$$
\begin{aligned}
&[S_a, S_b] = \mathrm{i}\varepsilon_{abc} S_c, \\
&[\eta_a, S_b] = \mathrm{i}\varepsilon_{abc} \eta_c \qquad \text{and} \qquad [\eta_a, \eta_b] = 0.
\end{aligned}
\tag{2}
$$

All indecomposable representations[3] of HG(1, 3) which, when restricted to the rotation subgroup, are decomposed to direct sums of vector and scalar representations, were found in [3]. These representations (denoted as $D(m, n, \lambda)$) are labeled by triplets of numbers: n, m and λ. These numbers take the values

$$
-1 \leqslant (n - m) \leqslant 2, \qquad n \leqslant 3, \qquad \lambda = \begin{cases} 0 & \text{if} \quad m = 0, \\ 1 & \text{if} \quad m = 2 \text{ or } n - m = 2, \\ 0, 1 & \text{if} \quad m = 1, \ n \neq 3. \end{cases}
\tag{3}
$$

In accordance with (3), there exist ten non-equivalent indecomposable representations $D(m, n, \lambda)$. Their carrier spaces can include three types of rotational scalars A, B, C and five types of vectors $\mathbf{R}$, $\mathbf{U}$, $\mathbf{W}$, $\mathbf{K}$, $\mathbf{N}$ whose transformation laws with respect to the Galilei boost are [10]

$$
\begin{aligned}
A &\to A' = A, \\
B &\to B' = B + \mathbf{v} \cdot \mathbf{R}, \\
C &\to C' = C + \mathbf{v} \cdot \mathbf{U} + \tfrac{1}{2}\mathbf{v}^2 A, \\
\mathbf{R} &\to \mathbf{R}' = \mathbf{R}, \\
\mathbf{U} &\to \mathbf{U}' = \mathbf{U} + \mathbf{v}A, \\
\mathbf{W} &\to \mathbf{W}' = \mathbf{W} + \mathbf{v} \times \mathbf{R}, \\
\mathbf{K} &\to \mathbf{K}' = \mathbf{K} + \mathbf{v} \times \mathbf{R} + \mathbf{v}A, \\
\mathbf{N} &\to \mathbf{N}' = \mathbf{N} + \mathbf{v} \times \mathbf{W} + \mathbf{v}B + \mathbf{v}(\mathbf{v} \cdot \mathbf{R}) - \tfrac{1}{2}\mathbf{v}^2 \mathbf{R},
\end{aligned}
\tag{4}
$$

where $\mathbf{v}$ is a vector whose components are parameters of the Galilei boost, $\mathbf{v} \cdot \mathbf{R}$ and $\mathbf{v} \times \mathbf{R}$ are scalar and vector products of vectors $\mathbf{v}$ and $\mathbf{R}$ respectively.

Carrier spaces of these indecomposable representations of the group HG(1, 3) include such sets of scalars A, B, C and vectors $\mathbf{R}$, $\mathbf{U}$, $\mathbf{W}$, $\mathbf{K}$, $\mathbf{N}$ which transform among themselves w.r.t. transformations (4) but cannot be split into a direct sum of invariant subspaces. There

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[3] Let us remind that a representation of a group $\mathfrak{G}$ in a normalized vector space $\mathfrak{C}$ is *irreducible* if its carrier space $\mathfrak{C}$ does not include subspaces invariant w.r.t. $\mathfrak{G}$. The representation is called *indecomposable* if $\mathfrak{C}$ does not include invariant subspaces $\mathfrak{C}_{\mathfrak{I}}$ which are orthogonal to $\mathfrak{C} \backslash \mathfrak{C}_{\mathfrak{I}}$. Irreducible representations are indecomposable too but indecomposable representations can be reducible in the sense that their carrier spaces can include (non-orthogonal) invariant subspaces.

exist exactly ten such sets which are listed in the following equation:

$$
\begin{aligned}
\{A\} &\iff D(0, 1, 0), \\
\{\mathbf{R}\} &\iff D(1, 0, 0), \\
\{B, \mathbf{R}\} &\iff D(1, 1, 0), \\
\{A, \mathbf{U}\} &\iff D(1, 1, 1), \\
\{A, \mathbf{U}, C\} &\iff D(1, 2, 1), \\
\{\mathbf{W}, \mathbf{R}\} &\iff D(2, 0, 0), \\
\{\mathbf{R}, \mathbf{W}, B\} &\iff D(2, 1, 0), \\
\{A, \mathbf{K}, \mathbf{R}\} &\iff D(2, 1, 1), \\
\{A, B, \mathbf{K}, \mathbf{R}\} &\iff D(2, 2, 1), \\
\{B, \mathbf{N}, \mathbf{W}, \mathbf{R}\} &\iff D(3, 1, 1).
\end{aligned}
\tag{5}
$$

Thus, in contrast to the relativistic case, where there are only three Lorentz covariant quantities which transform as vectors or scalars under rotations (i.e., a relativistic 4-vector, antisymmetric tensor of the second order and a scalar), there are ten indecomposable sets of the Galilei vectors and scalars which we have enumerated in equation (5).

3. Contractions of representations of the Lorentz algebra

It is well known that the Galilei algebra can be obtained from the Poincaré one by a limiting procedure called 'the Inönü–Wigner contraction' [9]. Representations of these algebras can also be connected by this kind of contraction. However, this connection is more complicated for two reasons. First, contraction of a non-trivial representation of the Lorentz algebra yields to the representation of the homogeneous Galilei algebra in which generators of the Galilei boosts are represented trivially, so that to obtain a non-trivial representation it is necessary to apply in addition a similarity transformation which depends on a contraction parameter in a tricky way. Second, to obtain indecomposable representations of hg(1, 3), it is necessary, in general, to start with completely reducible representations of the Lie algebra of the Lorentz group.

In paper [3], representations of the Lorentz group which can be contracted to representations $D(m, n, \lambda)$ of the Galilei group were found and the related contractions specified. Here, we present only a part of the results from [3] which will be used in what follows.

Let us begin with the representation $D\left(\frac{1}{2}, \frac{1}{2}\right)$ of the Lie algebra $so(1, 3)$ of the Lorentz group, whose carrier space is formed by 4-vectors. Basis of this representation is given by 4×4 matrices of the following form:

$$
S_{ab} = \varepsilon_{abc} \begin{pmatrix} s_c & \mathbf{0}_{3 \times 1} \\ \mathbf{0}_{1 \times 3} & 0 \end{pmatrix}, \qquad S_{0a} = \begin{pmatrix} \mathbf{0}_{3 \times 3} & -k_a^\dagger \\ k_a & 0 \end{pmatrix}.
\tag{6}
$$

Here, s_a are matrices of spin 1 with the elements $(s_a)_{bc} = i\varepsilon_{abc}$ and k_a are 1×3 matrices of the form

$$
k_1 = (i, 0, 0), \qquad k_2 = (0, i, 0), \qquad k_3 = (0, 0, i).
\tag{7}
$$

The Inönü–Wigner contraction consists of the transformation to a new basis

$$
S_{ab} \to S_{ab}, \qquad S_{0a} \to \varepsilon S_{0a}
$$

followed by a similarity transformation of all basis elements $S_{\mu\nu} \to S'_{\mu\nu} = V S_{\mu\nu} V^{-1}$ with a matrix V depending on a contraction parameter ε. Moreover, V depends on ε in such a way that all transformed generators S'_{ab} and $\varepsilon S'_{0a}$ are kept non-trivial and non-singular when $\varepsilon \to 0$ [9].

There exist two matrices V for representation (6), namely,

$$V_1 = \begin{pmatrix} \varepsilon I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 1 \end{pmatrix} \qquad \text{and} \qquad V_2 = \begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \varepsilon \end{pmatrix}. \tag{8}$$

Using V_1 we obtain

$$S'_{ab} = V_1 S_{ab} V_1^{-1} = S_{ab}, \qquad S'_{0a} = \varepsilon V_1 S_{0a} V_1^{-1} = \begin{pmatrix} \mathbf{0}_{3\times3} & -\varepsilon^2 k_a^\dagger \\ k_a & 0 \end{pmatrix}. \tag{9}$$

Then, passing ε to zero, we come to the following matrices:

$$S_a = \frac{1}{2}\varepsilon_{abc} S_{bc} = \begin{pmatrix} s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \qquad \eta_a = \lim S'_{0a}|_{\varepsilon\to0} = \begin{pmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ k_a & 0 \end{pmatrix}. \tag{10}$$

Analogously, using matrix V_2 we obtain

$$S_a = \begin{pmatrix} s_a & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}, \qquad \eta_a = \begin{pmatrix} \mathbf{0}_{3\times3} & -k_a^\dagger \\ \mathbf{0}_{1\times3} & 0 \end{pmatrix}. \tag{11}$$

Matrices (10) and (11) satisfy commutation relations (2), i.e., they realize representations of the algebra $hg(1, 3)$. More precisely, they form generators of the indecomposable representations $D(1, 1, 0)$ and $D(1, 1, 1)$ of the homogeneous Galilei group respectively. Indeed, denoting vectors from the related representation spaces as

$$\Psi = \begin{pmatrix} R_1 \\ R_2 \\ R_3 \\ B \end{pmatrix} \qquad \text{for} \quad D(1, 1, 0) \qquad \text{and} \qquad \tilde{\Psi} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ A \end{pmatrix} \qquad \text{for} \quad D(1, 1, 1)$$

and using the transformation laws (4) for $A, B, \mathbf{R} = \text{column}(R_1, R_2, R_3)$ and $\mathbf{U} = \text{column}(U_1, U_2, U_3)$ we easily find the corresponding Galilei boost generators η_a in the forms (10) and (11). As far as rotation generators S_a are concerned, they are direct sums of matrices of spin 1 (which are responsible for transformations of 3-vectors $\mathbf{R}$ and $\mathbf{U}$) and zero matrices (which keep scalars A and B invariant).

To obtain the five-dimensional representation $D(1, 2, 1)$, we have to start with a direct sum of the representations $D\left(\frac{1}{2}, \frac{1}{2}\right)$ and $D(0, 0)$ of the Lorentz group. The corresponding generators of the algebra $so(1, 3)$ have the form

$$\hat{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} & \cdot \\ \cdot & 0 \end{pmatrix}, \tag{12}$$

where $S_{\mu\nu}$ are matrices (6) and the dots denote zero matrices of appropriate dimensions. The matrix of the corresponding similarity transformation can be written as

$$V_3 = \begin{pmatrix} I_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & -\frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon \\ 0_{1\times3} & \varepsilon^{-1} & \varepsilon^{-1} \end{pmatrix}, \qquad V_3^{-1} = \begin{pmatrix} I_{3\times3} & 0_{3\times1} & 0_{3\times1} \\ 0_{1\times3} & -\varepsilon^{-1} & \frac{1}{2}\varepsilon \\ 0_{1\times3} & \varepsilon^{-1} & \frac{1}{2}\varepsilon \end{pmatrix}. \tag{13}$$

As a result, we obtain the following basis elements of the representation $D(1, 2, 1)$ of the algebra $hg(1, 3)$:

$$S_a = \begin{pmatrix} s_a & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{3\times1} & 0 & 0 \\ \mathbf{0}_{3\times1} & 0 & 0 \end{pmatrix}, \qquad \eta_a = \begin{pmatrix} \mathbf{0}_{3\times3} & k_a^\dagger & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & 0 & 0 \\ k_a & 0 & 0 \end{pmatrix}. \tag{14}$$

Matrices η_a in (14) generate transformations (4) of the components of vector-function $\hat{\Psi} = \mathtt{column}(\mathbf{U}, A, C)$ so that the relation

$$\hat{\Psi} \to \hat{\Psi}' = \exp(\mathrm{i}\boldsymbol{\eta} \cdot \boldsymbol{v})\hat{\Psi}$$

written componentwise, presents the transformation properties for $\mathbf{U}$, A and C written in (4).

Considering the representation $D(1, 0) \oplus D(0, 1)$ of the Lorentz group whose generators are 6×6 matrices

$$S_{ab} = \varepsilon_{abc}\begin{pmatrix} s_c & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & s_c \end{pmatrix} \qquad \text{and} \qquad S_{0a} = \begin{pmatrix} \mathbf{0}_{3\times3} & -s_a \\ s_a & \mathbf{0}_{3\times3} \end{pmatrix}, \qquad (15)$$

we have shown in [3] that it can be contracted only to one indecomposable representation of the HG(1, 3), namely, to the representation $D(2, 0, 0)$. The corresponding contraction matrix can be chosen in the following form [3]:

$$V_4 = \begin{pmatrix} \varepsilon I_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & I_{3\times3} \end{pmatrix}. \qquad (16)$$

In the present paper, we shall use also another contraction matrix:

$$V_5 = \begin{pmatrix} I_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \varepsilon I_{3\times3} \end{pmatrix}. \qquad (17)$$

Matrices V_4 and V_5 are unitary equivalent but can lead to different results when applied to search for Galilei limits of relativistic equations whose solutions form a carrier space of the representation $D(1, 0) \oplus D(0, 1)$ but also include additional dependent variables. It happens, e.g., when one considers Galilei limits of the Maxwell equations with currents and charges.

4. Galilei massless fields

For constructions of Galilei massless equations it is possible to use the same approach as in [11] where equations for the massive fields have been derived. However, here we prefer to apply another technique which consists in contractions of the appropriate relativistic wave equations.

4.1. Galilei limits of the Maxwell equations

According to the Lévy-Leblond and Le Bellac analysis from 1967 [1, 12] (see also [13, 14]) there are two Galilean limits of the Maxwell equations.

In the so-called 'magnetic' Galilean limit, we receive pre-Maxwellian electromagnetism. The corresponding equations for the magnetic field $\mathbf{H}$ and electric field $\mathbf{E}$ read

$$\nabla \times \mathbf{E}_m - \frac{\partial \mathbf{H}_m}{\partial t} = 0, \qquad \nabla \cdot \mathbf{E}_m = ej_m^0,$$
$$\nabla \times \mathbf{H}_m = e\mathbf{j}_m, \qquad \nabla \cdot \mathbf{H}_m = 0, \qquad (18)$$

where $j = \left(j_m^0, \mathbf{j}_m\right)$ is an electric current and e denotes an electric charge.

Equations (18) are invariant with respect to the Galilei transformations (1) provided vectors $\mathbf{H}_m, \mathbf{E}_m$ and electric current j co-transform as

$$\mathbf{H}_m \to \mathbf{H}_m, \qquad \mathbf{E}_m \to \mathbf{E}_m - \mathbf{v} \times \mathbf{H}_m,$$
$$\mathbf{j}_m \to \mathbf{j}_m, \qquad j_m^0 \to j_m^0 + \mathbf{v} \cdot \mathbf{j}_m. \qquad (19)$$

Introducing a Galilean vector-potential $A_m = (A^0, \mathbf{A})$ such that

$$\mathbf{H}_m = \nabla \times \mathbf{A}, \qquad \mathbf{E}_m = \frac{\partial \mathbf{A}}{\partial t} - \nabla A^0 \qquad (20)$$

we obtain from (19) the following transformation laws for A:

$$A^0 \to A^0 + \mathbf{v} \cdot \mathbf{A}, \qquad \mathbf{A} \to \mathbf{A}. \tag{21}$$

The other Galilean limit of the Maxwell equations, i.e., the 'electric' one looks as

$$\nabla \times \mathbf{H}_e + \frac{\partial \mathbf{E}_e}{\partial t} = e\mathbf{j}_e, \qquad \nabla \cdot \mathbf{E}_e = ej_e^4, \tag{22}$$
$$\nabla \times \mathbf{E}_e = 0, \qquad \nabla \cdot \mathbf{H}_e = 0,$$

with the Galilean transformation laws of the following form:

$$\mathbf{H}_e \to \mathbf{H}_e + \mathbf{v} \times \mathbf{E}_e, \qquad \mathbf{E}_e \to \mathbf{E}_e, \tag{23}$$
$$\mathbf{j}_e \to \mathbf{j}_e + \mathbf{v}j_e^4, \qquad j_e^4 \to j_e^4.$$

In contrast to j_m^0 in equation (19), the scalar component of current j_e^4 is not changed under the Galilei boost. In (22), (23) and in the following text, we use the upper index 4 to mark such vector components which are rotational and Galilean scalars, while the upper index 0 is reserved for components which are rotational scalars but are changed under the Galilei boost.

Vectors $\mathbf{H}_e$ and $\mathbf{E}_e$ can be expressed via vector potentials as

$$\mathbf{H}_e = \nabla \times \mathbf{A}, \qquad \mathbf{E}_e = -\nabla A^4, \tag{24}$$

with the corresponding Galilei transformations for the vector-potential

$$A^4 \to A^4, \qquad \mathbf{A} \to \mathbf{A} + \mathbf{v}A^4. \tag{25}$$

The Galilean limits of the Maxwell equations (found in [1, 12]) admit clear interpretation in the representation theory. The thing is that there are exactly two non-equivalent representations of the homogeneous Galilei group the carrier spaces of which are 4-vectors—the representations $D(1, 1, 0)$ and $D(1, 1, 1)$. In other words, there are exactly two non-equivalent Galilean transformation laws for 4-vector potentials and currents, which are given explicitly in equations (19), (21) and (23), (25), respectively. Equations for massless fields invariant with respect to these transformations are written in (18) and (22).

Let us note that both representations, i.e., $D(1, 1, 0)$ and $D(1, 1, 1)$, can be obtained via contractions of the representation $D(1/2, 1/2)$ of the Lorentz group whose carrier space is formed by relativistic 4-vectors. The related contraction matrices are written explicitly in (8). Each of these contractions generates a Galilei limit of the Maxwell equations either in the form (18) or (22). In sections 4.3 and 4.4, we shall obtain equations (18) and (22) via contraction of a more general system of relativistic equations for massless fields.

4.2. Extended Galilei electromagnetism

In accordance with our analysis of vector field representations of the Galilei group there exists only one representation, namely, $D(1, 2, 1)$ whose carrier space is formed by 5-vectors. Such 5-vectors appear naturally in many Galilean models, especially in those which are constructed via *reduction technique* [3], i.e., starting with models invariant with respect to the extended Poincaré group $P(1, 4)$ and then reducing them to be invariant w.r.t. its Galilei subgroup.

As mentioned in our paper [3], there are possibilities of introducing such different five-component gauge fields which join and extend the magnetic and electric Galilei limits of the considered relativistic 4-vector potentials. However, physical meanings of the corresponding theories have not been clarified. Moreover, as we have seen, the Maxwell electrodynamics can be contracted either to magnetic limit (18) or to the electric limit (22), and it has been generally accepted to think that it is impossible to formulate a consistent theory which includes both 'electric' and 'magnetic' Galilean gauge fields, (see, e.g., [5]).

Accepting the correctness of this statement, we shall nevertheless show that in some sense it is possible to join the 'electric' and 'magnetic' Galilei gauge fields, since the Galilean 5-vector potential appears naturally via contraction of a relativistic theory. Rather surprisingly, the corresponding relativistic equations are decoupled to two non-interacting subsystems, whereas their contracted counterparts appear to be coupled. This is in accordance with the observation presented in [3] that some indecomposable representations of the homogeneous Galilei group appear via contractions of completely reducible representations of the Lorentz group.

Let us begin with relativistic equations for vector-potential A^ν:

$$p^\mu p_\mu A^\nu = -ej^\nu \tag{26}$$

in the Lorentz gauge, i.e., fulfilling

$$p_\mu A^\mu = 0 \qquad \text{or} \quad p_0 A^0 = \mathbf{p} \cdot \mathbf{A}. \tag{27}$$

In equations (26) and (27), $p_\mu = i\frac{\partial}{\partial x^\mu}$, indices μ, ν run over the values μ, $\nu = 0, 1, 2, 3$, $\mathbf{A}$ and $\mathbf{p}$ are vector whose components are A^1, A^2, A^3 and p_1, p_2, p_3 correspondingly.

Let us consider in addition the inhomogeneous d'Alembert equation for a relativistic scalar field denoted as A^4:

$$p^\mu p_\mu A^4 = -ej^4. \tag{28}$$

Introducing the related vectors of the field strengths in the standard form

$$\mathbf{H} = \nabla \times \mathbf{A}, \qquad \mathbf{E} = \frac{\partial \mathbf{A}}{\partial x_0} - \nabla A^0, \qquad \mathbf{F} = -\nabla A^4, \qquad F^0 = \frac{\partial A^4}{\partial x_0}, \tag{29}$$

we get the Maxwell equations for $\mathbf{E}$ and $\mathbf{H}$:

$$\begin{aligned}
\nabla \times \mathbf{E} - \frac{\partial \mathbf{H}}{\partial x_0} = 0, &\qquad \nabla \cdot \mathbf{H} = 0, \\
\nabla \times \mathbf{H} + \frac{\partial \mathbf{E}}{\partial x_0} = e\mathbf{j}, &\qquad \nabla \cdot \mathbf{E} = ej^0
\end{aligned} \tag{30}$$

and the following equations for $\mathbf{F}$ and F^0:

$$\begin{aligned}
\frac{\partial F^0}{\partial x_0} - \nabla \cdot \mathbf{F} = ej^4, \\
\nabla \times \mathbf{F} = 0, \qquad \frac{\partial \mathbf{F}}{\partial x_0} = \nabla F^0.
\end{aligned} \tag{31}$$

Clearly, the system of equations (30) and (31) is completely decoupled. Its Galilean counterpart obtained using the Inönü–Wigner contraction appears to be, rather surprisingly, coupled. This contraction can be made directly for equations (30) and (31) but we prefer another, a more simple way with potential equations (26).

The system of equations (26)–(28) describes a decoupled system of relativistic equations for the five-component function:

$$A = \texttt{column}(A^1, A^2, A^3, A^0, A^4) = \texttt{column}(\mathbf{A}, A^0, A^4). \tag{32}$$

Moreover, the components (A^1, A^2, A^3, A^0) transform as a 4-vector and A^4 transforms as a scalar. The related generators (12) of the Lorentz group realize a direct sum of representations of the algebra $so(1,3)$, namely $D\left(\frac{1}{2}, \frac{1}{2}\right) \oplus D(0,0)$.

In accordance with [3], the completely reduced representation of the Lie algebra of the Lorentz group whose basis elements are given by equation (12) can be contracted either to a direct sum of indecomposable representations of the Galilei algebra hg(1, 3) or to indecomposable representation $D(1, 2, 1)$ of this algebra.

Let us consider the second possibility, i.e., a contraction to the indecomposable representation. Such contraction is presented in equations (12)–(14).

Let us demonstrate now that this contraction reduces the decoupled relativistic systems (26) and (28) to a system of the coupled equations invariant with respect to the Galilei group. Indeed, denoting $U_3 A = \texttt{column}(\mathbf{A}', A'^0, A'^4)$ and $U_3 j = \texttt{column}(\mathbf{j}' j'^0, j'^4)$ by A' and j' respectively and taking into account that a non-relativistic variable t is associated with the relativistic variable x_0 by $x_0 = ct \sim \frac{1}{\varepsilon} t$, we come to the following system of equations for the transformed quantities:

$$\mathbf{p}^2 A'^k = -e j'^k, \qquad \mathrm{i}\frac{\partial A'^4}{\partial t} = \mathbf{p} \cdot \mathbf{A}'. \tag{33}$$

Generators of the Galilei group for vectors A' and j' are expressed in equation (14), so under the Galilei transformations (1) they co-transform in accordance with the representation $D(1, 2, 1)$, i.e.,

$$A^0 \to A^0 + \mathbf{v} \cdot \mathbf{A} + \frac{\mathbf{v}^2}{2} A^4, \qquad \mathbf{A} \to \mathbf{A} + \mathbf{v} A^4, \qquad A^4 \to A^4, \tag{34}$$

and

$$j^4 \to j^4, \qquad \mathbf{j} \to \mathbf{j} + \mathbf{v} j^4, \qquad j^0 \to j^0 + \mathbf{v} \cdot \mathbf{j} + \tfrac{1}{2} v^2 j^4. \tag{35}$$

Of course, transformations (1), (34) and (35) keep system (33) invariant. In accordance with (33), the corresponding field strengths (compare with (29))

$$\mathbf{W} = \nabla \times \mathbf{A}', \qquad \mathbf{N} = \frac{\partial \mathbf{A}'}{\partial t} - \nabla A'^0, \qquad \mathbf{R} = \nabla A'^4, \qquad B = \frac{\partial A'^4}{\partial t} \tag{36}$$

satisfy the following equations:

$$\begin{aligned}
\mathcal{C} &\equiv \nabla \cdot \mathbf{N} - \frac{\partial}{\partial t} B - e j^0 = 0, \\
\mathcal{U} &\equiv \frac{\partial}{\partial t} \mathbf{R} + \nabla \times \mathbf{W} - e\mathbf{j} = 0, \\
\mathcal{A} &\equiv \nabla \cdot \mathbf{R} - e j^4 = 0, \\
\mathcal{N} &\equiv \frac{\partial}{\partial t} \mathbf{W} + \nabla \times \mathbf{N} = 0, \\
\mathcal{W} &\equiv \frac{\partial}{\partial t} \mathbf{R} - \nabla B = 0, \\
\mathcal{R} &\equiv -\nabla \times \mathbf{R} = 0, \qquad \text{and} \\
\mathcal{B} &\equiv \nabla \cdot \mathbf{W} = 0.
\end{aligned} \tag{37}$$

These equations are covariant with respect to the Galilei group like (33). Moreover, the Galilei transformations for fields $\mathbf{R}, \mathbf{W}, \mathbf{N}$, B and current j are given by equations (4) and (35) correspondingly. In other words, these fields and current j form carrier spaces of the representations $D(3, 1, 1)$ and $D(1, 2, 1)$ of the algebra $\mathrm{hg}(1, 3)$, respectively.

In contrast to a decoupled relativistic system of equations (30) and (31), its Galilei counterpart (37) appears to be a coupled system of equations for vectors $\mathbf{R}, \mathbf{W}, \mathbf{N}$ and scalar B.

The system of equations equivalent to (37) was derived in paper [4] via reduction of generalized Maxwell equations invariant with respect to the extended Poincaré group $P(1, 4)$ with one time and four spatial variables. We have proved that this system is nothing else than a contracted version of systems (30) and (31), including the ordinary Maxwell equations and equations for a four gradients of the scalar potential. It other words, the system of the Galilei invariant equations (37) admits a clear physical interpretation as a non-relativistic limit of the system of familiar equations (30) and (31).

4.3. Reduced Galilean electromagnetism

In contrast to the relativistic case, the Galilei invariant approach allows to reduce the number of field variables. For example, considering the magnetic limit (18) of the Maxwell equations it is possible to restrict ourselves to the case $\mathbf{H}_m = 0$, since this condition is invariant with respect to the Galilei transformations due to (19). Note that in a relativistic theory such condition can be only imposed in a particular frame of reference and will be violated by the Lorentz transformations.

In the mentioned sense, equations (37) are reducible too. They are defined on the most extended multiplet of vector and spinor fields which is a carrier space of an indecomposable representation of the homogeneous Galilei group. The corresponding representation $D(3, 1, 1)$ is indecomposable but reducible, i.e., it includes subspaces invariant w.r.t. the Galilei group. This makes possible to reduce a number of dependent components of equations (37) without violating its Galilei invariance.

In this section, we shall consider systematically all possible Galilei invariant constrains which can be imposed on solutions of equations (37) and present the corresponding reduced versions of the Galilean electromagnetism.

In accordance with (4) and (35), vector $\mathbf{R}$ and the fourth component j^4 of the current form invariant subspaces w.r.t. the Galilei transformations. Thus we can impose the Galilei-invariant conditions

$$\mathbf{R} = 0 \qquad \text{or} \quad \nabla A^4 = 0, \qquad j^4 = 0, \tag{38}$$

and reduce system (37) to the following one:

$$\begin{aligned}
&\frac{\partial}{\partial t}\tilde{\mathbf{H}} + \nabla \times \tilde{\mathbf{E}} = 0, \\
&\nabla \times \tilde{\mathbf{H}} = e\mathbf{j}, \qquad \nabla \cdot \tilde{\mathbf{H}} = 0, \\
&\nabla \cdot \tilde{\mathbf{E}} = \frac{\partial}{\partial t}S + ej^0, \\
&\nabla S = 0,
\end{aligned} \tag{39}$$

where we have used notations $\tilde{\mathbf{H}} = \mathbf{W}|_{\mathbf{R}\equiv 0}$, $\tilde{\mathbf{E}} = \mathbf{N}|_{\mathbf{R}\equiv 0}$ and $S = B|_{\mathbf{R}\equiv 0}$.

Vectors $\tilde{\mathbf{E}}$, $\tilde{\mathbf{H}}$ and scalar S belong to a carrier space of the representation $D(2, 1, 1)$. Their Galilei transformation laws are

$$\tilde{\mathbf{E}} \to \tilde{\mathbf{E}} + \mathbf{v} \times \tilde{\mathbf{H}} + \mathbf{v}S, \qquad \tilde{\mathbf{H}} \to \tilde{\mathbf{H}}, \quad S \to S. \tag{40}$$

In accordance with (40), S belongs to an invariant subspace of the Galilei transformations, so we can impose the following additional Galilei-invariant condition:

$$S = 0 \qquad \text{or} \quad \frac{\partial A^4}{\partial t} = 0. \tag{41}$$

As a result, we come to equations (18), i.e., to the magnetic limit of Maxwell's equations. Thus equations (18) are nothing else than the system of equations (37) with the additional Galilei-invariant constrains (38) and (41).

The Galilei transformations of solutions of equations (18) are given by formulae (19). Again, we recognize an invariant subspace spanned on vectors $\mathbf{H}_m$, and thus it is possible to impose the invariant condition

$$\mathbf{H}_m = 0 \qquad \text{or} \quad \mathbf{A} = \nabla\varphi, \qquad \mathbf{j}_m = 0, \tag{42}$$

where φ is a solution of the Laplace equation. As a result, we come to the following system:

$$\nabla \times \widehat{\mathbf{E}} = 0, \qquad \nabla \cdot \widehat{\mathbf{E}} = e\rho, \tag{43}$$

where we have used notations $\widehat{\mathbf{E}} = \mathbf{E}_m|_{\mathbf{H}_m\equiv 0}$ and $\rho = j_m^0\big|_{\mathbf{H}_m\equiv 0}$.

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Equation (43) remains still Galilei-invariant, since both $\widehat{\mathbf{E}}$ and ρ are not changed under the Galilei transformations. The corresponding potential $\tilde{A}$ is constrained by conditions (38), (41) and (42). Moreover, up to gauge transformations, it is possible to set $\mathbf{A} = 0$ in (42).

4.4. Other reductions

Equations (39), (18) and (43) exhaust all Galilei-invariant systems which can be obtained starting with (37) and imposing additional constraints which reduce the number of dependent variables. To find the other Galilei-invariant equations for massless vector fields, we shall use the observation that the Galilei transformations of equations (37) have the form written in relations (4) if we replace here capital letters by calligraphic ones, i.e., $\mathbf{N} \to \mathcal{N}$, $\mathbf{W} \to \mathcal{W}$, ..., Thus, the following subsystem of equations (37) (obtained by excluding a self-invariant pair of equations, namely, $\mathcal{N} = 0$ and $\mathcal{C} = 0$):

$$
\begin{aligned}
\mathcal{U} &\equiv \nabla \times \mathbf{W} + \frac{\partial}{\partial t}\mathbf{R} - e\mathbf{j} = 0, \\
\mathcal{A} &\equiv \nabla \cdot \mathbf{R} - ej^4 = 0, \\
\mathcal{W} &\equiv \frac{\partial}{\partial t}\mathbf{R} - \nabla B = 0, \\
-\mathcal{R} &\equiv \nabla \times \mathbf{R} = 0, \\
\mathcal{B} &\equiv \nabla \cdot \mathbf{W} = 0
\end{aligned}
\tag{44}
$$

is Galilei covariant too and does not include dependent variables $\mathbf{N}$ and j^0. The Galilei transformations of $\mathbf{W}$, $\mathbf{R}$, B and $\mathbf{j}$, j^4 remain determined by equations (4) and (35).

Following analogous reasonings, it is possible to exclude from (44) the equations $\mathcal{U} = 0$ and $\mathcal{B} = 0$ and to obtain the system

$$
\begin{aligned}
\nabla \cdot \mathbf{R} - ej^4 &= 0, \\
\frac{\partial}{\partial t}\mathbf{R} - \nabla B &= 0, \\
\nabla \times \mathbf{R} &= 0,
\end{aligned}
\tag{45}
$$

which includes only two vector and two scalar variables. The corresponding potential without loss of generality reduces to the only variable A^4.

The other possibility of reducing system (44) consists of excluding the equation $\mathcal{W} = 0$. As a result, we come to the electric limit for the Maxwell equations (22) for $\mathbf{W} = \mathbf{H}_e$ and $\mathbf{R} = \mathbf{E}_e$.

Thus, in addition to (18), (37), (39) and (43), we have three other Galilei-invariant systems given by equations (22), (44) and (45). These equations admit additional reductions by imposing Galilei-invariant constrains to their solutions.

Considering (22), we easily find that another possible invariant condition is formed by the pair $\mathbf{E}_e = 0$, $j_e^4 = 0$ so that (22) reduces to the following equations:

$$
\nabla \times \hat{\mathbf{H}} = e\mathbf{j}, \qquad \nabla \cdot \hat{\mathbf{H}} = 0,
\tag{46}
$$

where $\hat{\mathbf{H}}$ denotes $\mathbf{H}_e|_{\mathbf{E}_e \equiv 0}$.

Let us return to system (44). This system can be reduced by imposing the Galilei-invariant pair of conditions $\mathbf{R} = 0$, $j^4 = 0$ to the following (decoupled) form:

$$
\begin{aligned}
\nabla \times \hat{\mathbf{H}} - e\mathbf{j} &= 0, \\
\nabla \cdot \hat{\mathbf{H}} = 0, \qquad \nabla S &= 0,
\end{aligned}
\tag{47}
$$

where we have changed the notations $\mathbf{W} \to \hat{\mathbf{H}}$ and $B \to S$. The corresponding potential reads $A = (A^4, 0, \mathbf{A})$, where A^4 should satisfy the condition $\nabla A^4 = 0$.

We see that, in contrast to a relativistic theory, there exist a big variety of equations for massless vector fields invariant w.r.t. the Galilei group. The list of such equations is given by formulae (18), (22), (37), (39) and (43)–(47).

5. Nonlinear equations for vector fields

Starting with indecomposable representations of the group HG(1, 3) found in [3, 10], it is possible to find out various classes of partial differential equations invariant w.r.t. the Galilei group. In the previous sections, we have restricted ourselves to linear Galilean equations for vector and scalar fields and now we shall present nonlinear equations. More precisely, we shall study systems of quasilinear first-order equations invariant w.r.t. the representations discussed in section 2.

5.1. Galilei electromagnetic field in media

Let us consider the Maxwell equations for the electromagnetic field in a medium

$$
\begin{aligned}
\frac{\partial \mathbf{D}}{\partial t} &= \nabla \times \mathbf{H}, &\qquad \nabla \cdot \mathbf{D} &= 0, \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}, &\qquad \nabla \cdot \mathbf{B} &= 0.
\end{aligned}
\tag{48}
$$

Here $\mathbf{E}$ and $\mathbf{H}$ are vectors of the electric and magnetic field strengths and $\mathbf{D}$ and $\mathbf{B}$ denote the corresponding vectors of the electric and magnetic inductions. System (48) is underdetermined and has to be completed by constitutive equations which represent the medium properties. The simplest constitutive equations correspond to the case where $\mathbf{B}$ and $\mathbf{D}$ are proportional to $\mathbf{H}$ and $\mathbf{E}$, respectively, i.e.,

$$
\mathbf{B} = \mu \mathbf{H} \qquad \text{and} \qquad \mathbf{D} = \kappa \mathbf{E}.
\tag{49}
$$

Here μ and κ are constants.

In general, μ and κ can be scalar functions of $\mathbf{E}$ and $\mathbf{H}$ so that the related theories are essentially nonlinear. There are even more complex constitutive equations, e.g.,

$$
\mathbf{B} = \mu \mathbf{H} + \nu \mathbf{E}, \qquad \mathbf{D} = \kappa \mathbf{E} + \lambda \mathbf{H},
\tag{50}
$$

where μ, ν, κ and λ are some functions of $\mathbf{H}$ and $\mathbf{E}$. A popular example of the constitutive equations is the Born–Infeld system [15] which we consider in the following section.

Let us note that system (48) by itself, i.e., without constitutive equations, is invariant w.r.t. a very extended group which includes both the Poincaré and the Galilei groups as subgroups [16]. And just constitutive equations, e.g., (49) or (50), reduce this group to the Poincaré group.

Since we are studying Galilean aspects of the electrodynamics, it is naturally to pose a problem, weather there exist such constitutive equations which reduce the symmetry of system (48) to the Galilei group.

For this purpose, we shall search for Galilei-invariant constitutive equations in the form (50). Such equations are Galilei-invariant provided μ, ν, κ and λ are invariants of Galilei transformations and, in addition,

$$
\sigma \kappa = \nu, \qquad \mu = \sigma \lambda,
\tag{51}
$$

where σ is an invariant of the Galilei group.

The Galilei transformations of vectors $\mathbf{H}$, $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{B}$, which keep equations (48) invariant, have the form

$$\mathbf{E} \to \mathbf{E} + \mathbf{v} \times \mathbf{B}, \qquad \mathbf{H} \to \mathbf{H} - \mathbf{v} \times \mathbf{D}, \quad \mathbf{D} \to \mathbf{D}, \quad \mathbf{B} \to \mathbf{B}. \tag{52}$$

A list of independent bilinear invariants of these transformations reads

$$\mathbf{E} \cdot \mathbf{B}, \quad \mathbf{H} \cdot \mathbf{D}, \quad \mathbf{D}^2, \quad \mathbf{B}^2, \quad \mathbf{E} \cdot \mathbf{D} - \mathbf{H} \cdot \mathbf{B}. \tag{53}$$

Note that all the other invariants are their functions.

Thus, we have found the general Galilei-invariant equations for an electromagnetic field in media in the form (48) with constitutive equations (50), where μ, ν, κ and λ are arbitrary functions of invariants (53) satisfying conditions (51). In the following section, we shall present Galilean versions of the Born–Infeld equations.

5.2. Galilean Born–Infeld equations

The relativistic Born–Infeld equations include system (48) and the constitutive equations

$$\mathbf{D} = \frac{1}{L}(\mathbf{E} + (\mathbf{B} \cdot \mathbf{E})\mathbf{B}), \qquad \mathbf{H} = \frac{1}{L}(\mathbf{B} - (\mathbf{B} \cdot \mathbf{E})\mathbf{E}), \tag{54}$$

where $L = (1 + \mathbf{B}^2 - \mathbf{E}^2 - \mathbf{B} \cdot \mathbf{E})^{1/2}$. Equations (48) are Lorentz invariant. To figure out the corresponding representation of the Lorentz group explicitly, we represent vectors of its carrier space in the following form:

$$\Psi = \texttt{column}(\mathbf{B}, \mathbf{E}, \mathbf{D}, \mathbf{H}). \tag{55}$$

Then the associated generators of the Lorentz group are written as a direct sum of matrices (15), i.e., as

$$\hat{S}_{ab} = \begin{pmatrix} S_{ab} & 0 \\ 0 & S_{ab} \end{pmatrix}, \qquad S_{0a} = \begin{pmatrix} S_{0a} & 0 \\ 0 & S_{0a} \end{pmatrix}. \tag{56}$$

The Inönü–Wigner contraction can be found by using direct sums of the contracting matrices (10), i.e., by

$$V_6 = \begin{pmatrix} V_4 & \mathbf{0}_{6\times 6} \\ \mathbf{0}_{6\times 6} & V_5 \end{pmatrix} \qquad \text{or} \qquad V_7 = \begin{pmatrix} V_5 & \mathbf{0}_{6\times 6} \\ \mathbf{0}_{6\times 6} & V_4 \end{pmatrix}. \tag{57}$$

First, let us apply the contraction matrix V_6 on Ψ defined in (55). Then the vectors in Ψ will be transformed in such a way that $\Psi \to \Psi' = \texttt{column}(\mathbf{B}', \mathbf{E}', \mathbf{D}', \mathbf{H}') = \varepsilon V_6 \Psi$, with

$$\mathbf{E} = \mathbf{E}', \quad \mathbf{B} = \varepsilon \mathbf{B}', \quad \mathbf{D} = \mathbf{D}', \quad \mathbf{H} = \varepsilon \mathbf{H}'. \tag{58}$$

Substituting (58) into (48) and (54), taking into account that at the same time $\frac{\partial}{\partial x_0} \to \varepsilon \frac{\partial}{\partial t}$, $\nabla \to \nabla$ and equating terms with the lowest powers of ε, we come to the following system:

$$\frac{\partial \mathbf{D}'}{\partial t} = \nabla \times \mathbf{H}', \qquad \nabla \cdot \mathbf{D}' = 0,$$
$$\nabla \times \mathbf{E}' = 0, \qquad \nabla \cdot \mathbf{B}' = 0, \tag{59}$$

with the constitutive equations

$$\mathbf{D}' = \frac{\mathbf{E}'}{\sqrt{1 - \mathbf{E}'^2}}, \qquad \mathbf{H}' = \frac{\mathbf{B}' - (\mathbf{B}' \cdot \mathbf{E}')\mathbf{E}'}{\sqrt{1 - \mathbf{E}'^2}}. \tag{60}$$

Equations (59) and (60) are Galilei invariant. Moreover, under Galilei boosts, vectors $\mathbf{D}'$, $\mathbf{H}'$, $\mathbf{B}'$ and $\mathbf{E}'$ co-transform as

$$\begin{aligned}
\mathbf{D}' &\to \mathbf{D}', & \mathbf{H}' &\to \mathbf{H}' + \mathbf{v} \times \mathbf{D}', \\
\mathbf{B}' &\to \mathbf{B}' + \mathbf{v} \times \mathbf{E}', & \mathbf{E}' &\to \mathbf{E}'.
\end{aligned} \tag{61}$$

Analogously, starting again with (48) but using the contraction matrix V_7 instead of V_6, we obtain the following system of the Galilei-invariant equations:

$$\begin{aligned}
\nabla \times \mathbf{H}' &= 0, & \nabla \cdot \mathbf{D}' &= 0, \\
\frac{\partial \mathbf{B}'}{\partial t} &= -\nabla \times \mathbf{E}', & \nabla \cdot \mathbf{B}' &= 0,
\end{aligned} \tag{62}$$

which are supplemented with the Galilei-invariant constitutive equations

$$\mathbf{D}' = \frac{\mathbf{E}' + (\mathbf{B}' \cdot \mathbf{E}')\mathbf{B}'}{\sqrt{1 + \mathbf{B}'^2}}, \qquad \mathbf{H}' = \frac{\mathbf{B}'}{\sqrt{1 + \mathbf{B}'^2}}. \tag{63}$$

The corresponding transformation laws read

$$\begin{aligned}
\mathbf{D}' &\to \mathbf{D}' - \mathbf{v} \times \mathbf{H}', & \mathbf{H}' &\to \mathbf{H}', \\
\mathbf{B}' &\to \mathbf{B}', & \mathbf{E}' &\to \mathbf{E}' - \mathbf{v} \times \mathbf{B}'.
\end{aligned} \tag{64}$$

Thus we have seen that there exist two Galilei limits for the Maxwell equations in various media which are given by equations (59), (60) and (62), (63). Let us remark that the other mathematically possible Galilean limits of the Born–Infeld equations (for instance, by direct sums of two contracting matrices $V_5 \oplus V_5$ or $V_4 \oplus V_4$) yield trivial constitutive equations.

5.3. Quasilinear wave equations and Galilean Chern–Simons models

In this section, we present nonlinear terms (depending on vectors $\mathbf{RW}$, $\mathbf{N}$ and scalar B) which can be added to system (37) without violating its Galilei invariance. We restrict ourselves to linear and bilinear combinations of these vector and scalar components.

Using Galilean transformation laws (4), one can verify that the following scalars $\hat{j}_0$, $\hat{j}_4$ and vector $\mathbf{j}$:

$$\begin{aligned}
\hat{j}^0 &= \nu \mathbf{W} \cdot \mathbf{N} + \lambda \mathbf{R} \cdot \mathbf{W} + \sigma(B^2 - \mathbf{R} \cdot \mathbf{N}) + \omega \mathbf{R}^2 + \mu B, \\
\hat{\mathbf{j}} &= \nu(B\mathbf{W} + \mathbf{R} \times \mathbf{N}) + \sigma(\mathbf{R} \times \mathbf{W} + B\mathbf{R}) + \mu \mathbf{R}, \\
\hat{j}^4 &= \nu \mathbf{R} \cdot \mathbf{W} + \sigma \mathbf{R}^2
\end{aligned} \tag{65}$$

(where Greek letters denote arbitrary parameters) transform as components of a 5-vector from the carrier space of the representation $D(1, 2, 1)$ of the HG(1, 3). In other words, their transformation laws are given by relations (35) where, however, the 'hats' are absent.

It follows from the above that Galilei invariance of system (37) will not be violated if we add the terms $\hat{j}^0$, $\hat{\mathbf{j}}$ and $\hat{j}^4$ to the first, second and third equations of system (37), respectively. In addition, it is possible to add terms proportional to $\mathbf{N}$, $\mathbf{W}$, $\mathbf{R}$ and B to (4)–(7) equations

correspondingly. As a result, we obtain the following system:

$$\frac{\partial}{\partial t}B - \nabla \cdot \mathbf{N} + \nu \mathbf{W} \cdot \mathbf{N} + \lambda \mathbf{R} \cdot \mathbf{W} + \sigma(B^2 - \mathbf{R} \cdot \mathbf{N}) + \omega \mathbf{R}^2 + \mu B = ej^0,$$

$$\frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{W} + \nu(B\mathbf{W} + \mathbf{R} \times \mathbf{N}) + \sigma(\mathbf{R} \times \mathbf{W} + B\mathbf{R}) + \mu \mathbf{R} = e\mathbf{j},$$

$$\nabla \cdot \mathbf{R} + \nu \mathbf{R} \cdot \mathbf{W} + \sigma \mathbf{R}^2 = ej^4,$$

$$\frac{\partial}{\partial t}\mathbf{W} + \nabla \times \mathbf{N} + \rho \mathbf{N} = 0, \tag{66}$$

$$\frac{\partial}{\partial t}\mathbf{R} - \nabla B + \rho \mathbf{W} = 0,$$

$$-\nabla \times \mathbf{R} + \rho \mathbf{R} = 0, \qquad \text{and}$$

$$\nabla \cdot \mathbf{W} + \rho B = 0.$$

Formula (66) presents the most general Galilei-invariant quasilinear system which can be obtained from (37) by adding linear terms and quadratic nonlinearities. Let us consider in more detail a particular case of system (66) which corresponds to the zero values of arbitrary parameters ω, σ, λ, μ and ρ:

$$\frac{\partial}{\partial t}B - \nabla \cdot \mathbf{N} + \nu \mathbf{W} \cdot \mathbf{N} = ej^0,$$

$$\frac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{W} + \nu(B\mathbf{W} + \mathbf{R} \times \mathbf{N}) = e\mathbf{j},$$

$$\nabla \cdot \mathbf{R} + \nu \mathbf{R} \cdot \mathbf{W} = ej^4,$$

$$\frac{\partial}{\partial t}\mathbf{W} + \nabla \times \mathbf{N} = 0, \tag{67}$$

$$\frac{\partial}{\partial t}\mathbf{R} - \nabla B = 0,$$

$$\nabla \times \mathbf{R} = 0, \qquad \text{and}$$

$$\nabla \cdot \mathbf{W} = 0.$$

Let us note that in the case $\nu = 0$, equations (67) coincide with equations (36) describing 'extended Galilei electromagnetism'; see section 4.2. Thus the vectors $\mathbf{N}$, $\mathbf{W}$, $\mathbf{R}$ and scalar B can be expressed via derivatives of a 5-vector potential $A = (A^0, \mathbf{A}, A^4)$, see (29).

Starting with (67) and making reductions analogous to ones considered in subsections 4.3 and 4.4, it is easy to find reduced versions of this system. Let us present here the magnetic and electric limits of equations (67):

$$\nabla \cdot \mathbf{E}_m + \nu \mathbf{H}_m \cdot \mathbf{E}_m = ej^0, \qquad \nabla \times \mathbf{H}_m = e\mathbf{j},$$

$$\frac{\partial \mathbf{H}_m}{\partial t} - \nabla \times \mathbf{E}_m = 0, \qquad \nabla \cdot \mathbf{H}_m = 0 \tag{68}$$

and

$$\frac{\partial \mathbf{E}_e}{\partial t} + \nabla \times \mathbf{H}_e + \mu \mathbf{E_e} = e\mathbf{j}, \qquad \nabla \times \mathbf{E}_e = 0,$$

$$\nabla \cdot \mathbf{E}_e + \nu \mathbf{E}_e \cdot \mathbf{H}_e = ej^4, \qquad \nabla \cdot \mathbf{H}_e = 0. \tag{69}$$

It is necessary to stress that equations (67) admit a Lagrangian formulation. The corresponding Lagrangian reads

$$L = \tfrac{1}{2}(B^2 - \mathbf{W}^2) - \mathbf{N} \cdot \mathbf{R} + \nu(A^4 \mathbf{W} \cdot \mathbf{N} + A^0 \mathbf{R} \cdot \mathbf{W} - \mathbf{A} \cdot (\mathbf{W}B + \mathbf{R} \times \mathbf{N}))$$

$$- e(A^4 j^0 + A^0 j^4 - \mathbf{A} \cdot \mathbf{j}). \tag{70}$$

Lagrangian (70) includes products of potential A and field strengths and in this aspect can be treated as a Galilean version of the Chern–Simons Lagrangian [18]. On the other hand, system (67) is nothing but a Galilei-invariant analogue of the Carroll–Field–Jackiw (CFJ) model [19]. This model was formulated with a view to examine the possibility of Lorentz and CPT violations in Maxwell's electrodynamics and is invariant neither w.r.t. the Lorentz nor w.r.t. the Galilei transformations. Equations (67) can be derived via contraction of a generalized CFJ model which will be shown elsewhere.

Lagrangian (70) is invariant with respect to the Galilei group which includes in particular shifts of time and spatial variables. Thus in the case $j^0 = j^4 = 0$ and $\mathbf{j} = 0$, we can find the related energy–momenta tensor whose components are given in the following equation:

$$
\begin{aligned}
T^0{}_0 &= \tfrac{1}{2}(B^2 + W_b W_b), \\
T^0{}_a &= \varepsilon_{abc} N_b W_c - B N_a, \\
T^a{}_0 &= B R_a + \varepsilon_{abc} R_b W_c, \\
T^a{}_b &= N_a R_b + N_b R_a - W_a W_b + \delta_{ab}\left(\tfrac{1}{2}(B^2 + W_n W_n) - R_n W_n\right).
\end{aligned}
\tag{71}
$$

Tensors (71) satisfy the continuity equations

$$
\frac{\partial}{\partial t} T^\nu{}_0 + \frac{\partial}{\partial x_a} T^\nu{}_a = 0, \qquad \nu = 0, 1, 2, 3
\tag{72}
$$

and so generate conserved quantities. Moreover, the energy density $\mathcal{E}$ and momentum density $\mathcal{P}$ for a system described by equations (67) with $e = 0$ are associated with $T^0{}_0$ and $T^0{}_a$ and so can be written in the following form:

$$
\mathcal{E} = \tfrac{1}{2}(B^2 + \mathbf{W}^2), \qquad \mathcal{P} = \mathbf{N} \times \mathbf{W} - B\mathbf{N}.
\tag{73}
$$

It is interesting to note that the energy–momenta tensor (71) is valid also for the linear version of system (67), i.e., when $e = \nu = 0$. In other words, like in (1+2)-dimensional Chern–Simons model the 'interaction' terms with a coupling constant ν do not affect the energy–momenta tensor. This fact gives one more argument to specify (70) as a Galilean Chern–Simons Lagrangian.

Starting with Lagrangian (67) and using the Noether theorem, it is possible to specify conserved currents which correspond to other symmetries, i.e., the rotational and Galilei boosts ones. We reserve these possibilities for a future publication where the Galilei Chern–Simons system will be considered in more detail.

6. Discussion

The revision of classical results [1] associated with Galilean electromagnetism done in the present paper appears to be possible due to our knowledge of indecomposable representations of the homogeneous Galilei group defined on vector and scalar fields [3]. Thus the present paper completes the results of Le Bellac and Lévy-Leblond in [1] and presents an extended class of the Galilei-invariant equations for massless fields. Among them are decoupled systems of the first-order equations which include the same number of components as the Maxwell equations as well as equations with other numbers of components. The most extended system includes ten components, while the most reduced one only three.

It is necessary to stress that the majority of the obtained equations admit clear physical interpretations. For instance equations (43) and (46) are basic for electro- and magnetostatics, respectively. Our procedure of deducing the Galilei-invariant equations for vector fields used

in the present paper makes interpretations of the equations rather straightforward, since any obtained equation has its relativistic counterpart.

We see that a number of the Galilei wave equations for massless vector fields is rather huge, and so there are many possibilities of describing an interaction of non-relativistic charged particles with external gauge fields. Some of these possibilities have been discussed in [3, 11] (se also [4, 5, 8, 17]). Starting with the found equations and using the list of functional invariants for the Galilean vector fields presented in [10], it is easy to construct nonlinear models invariant with respect to the Galilei group, including its supersymmetric extensions. Some examples of nonlinear models have been discussed in section 5. In particular, Galilean versions of Born–Infeld and Chern–Simons systems are deduced here.

Let us note that in the case $\nu = e = 0$ the Lagrangian (70) can be reduced to the massless field part of the Lagrangian discussed in paper [6]. Our contribution is a demonstration how the related Euler–Lagrange equations (67) can be obtained via contraction of a relativistic system and a discussion of the related conservation laws. In particular, we show that the nonlinear interaction terms in (67) do not affect the energy–momenta tensor (71) which is valid for the linear system (36). Moreover, we have presented a much more general Galilei-invariant nonlinear system (66) which in principle cannot be obtained within *reduction approach* used in [6], i.e., starting with systems invariant w.r.t. the extended Poincaré group $P(1, 4)$ (group of motions of the flat (1+4)-dimensional Minkowski space) and then reducing them to Galilei-invariant systems.

The main result presented in this paper is a complete description of all linear first-order Galilei invariant equations for massless vector and scalar fields. Equations which can be obtained via contractions of relativistic systems are enumerated in subsections 4.2–4.4. In fact, we give also the most general description of Galilean first-order systems for vector and scalar fields, since we find all covariant differential forms for such fields. A complete list of these forms is given in the appendix.

In addition, we present an extended class of nonlinear Galilean systems which are discussed in section 5.

Acknowledgments

Appendix. Covariant differential forms

To complete our analysis of Galilei-invariant linear wave equations for vector and scalar fields, we present a full list of first-order differential forms which transform as indecomposable vectors sets under the Galilei transformations. In this way, we describe general linear Galilean equations of first order for scalar and vector fields.

Using exact transformation laws given by equations (1) and (4), it is not difficult to find the corresponding transformations for derivatives of vector fields. The differential operators $\frac{\partial}{\partial t}$ and ∇ transform as components of 4-vector from a carrier space of the representation $D(1, 1, 0)$ of the HG$(1, 3)$, thus to describe transformation properties of these derivatives it is sufficient to describe tensor products of this representation with all representations enumerated in equation (5). It is evident that the derivatives of vector fields can transform as scalars, vectors or second rank tensors under rotations. Restricting ourselves to those forms which transform as vectors or scalars, we obtain the following indecomposable sets of them:

for $D(0, 1, 0)$: $\{\mathcal{R}_1 = \nabla A\}$;

for $D(1, 0, 0)$: $\{\mathcal{R}_2 = -\nabla \times \mathbf{R}\}$ and $\{\mathcal{A}_1 = \nabla \cdot \mathbf{R}\}$

for $D(1, 1, 0)$: $\left\{\mathcal{R}_2, \mathcal{W}_2 = \dfrac{\partial \mathbf{R}}{\partial t} - \nabla B\right\} \supset \{\mathcal{R}_2\}$,   and $\{\mathcal{A}_1\}$;

for $D(1, 1, 1)$: $\left\{\mathcal{B}_1 = \dfrac{1}{2}\left(\dfrac{\partial A}{\partial t} + \nabla \cdot \mathbf{U}\right), \mathcal{W}_1 = \nabla \times \mathbf{U}, \mathcal{R}_1\right\} \supset (\{\mathcal{B}_1, \mathcal{R}_1\},$

$\{\mathcal{W}_1, \mathcal{R}_1\}, \{\mathcal{R}_1\})$,   and $\left\{\mathcal{A}_2 = \dfrac{\partial A}{\partial t} - \nabla \cdot \mathbf{U}\right\}$;

for $D(2, 0, 0)$: $\left\{\mathcal{U}_1 = \dfrac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{W}, \mathcal{A}_1\right\} \supset \{\mathcal{A}_1\}$,

and $\{\mathcal{B}_2 = \nabla \cdot \mathbf{W}, \mathcal{R}_2\} \supset \{\mathcal{R}_2\}$;

for $D(1, 2, 1)$: $\left\{\mathcal{N}_1 = \dfrac{\partial \mathbf{U}}{\partial t} - \nabla C, \mathcal{W}_1, \mathcal{R}_1, \widetilde{\mathcal{B}}_1 = \dfrac{\partial A}{\partial t}\right\} \supset \left\{\mathcal{W}_1, \mathcal{R}_1, \widetilde{\mathcal{B}}_1 = \dfrac{\partial A}{\partial t}\right\}$

$\supset (\{\widetilde{\mathcal{B}}_1, \mathcal{R}_1\}, \{\mathcal{W}_1, \mathcal{R}_1\}, \{\mathcal{R}_1\})$,   and $\{\mathcal{A}_2\}$;

for $D(2, 1, 0)$: $\{\mathcal{W}_2, \mathcal{R}_2, \mathcal{B}_2\} \supset (\{\mathcal{B}_2, \mathcal{R}_2\}, \{\mathcal{W}_2, \mathcal{R}_2\}, \{\mathcal{R}_2\})$,   and $\{\mathcal{A}_1\}$;

for $D(2, 1, 1)$: $\left\{\mathcal{K}_1 = \dfrac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{K}, \widetilde{\mathcal{R}}_1 = -\nabla A, \mathcal{A}_1\right\} \supset (\{\widetilde{\mathcal{R}}_1\}, \{\mathcal{A}_1\})$

and $\left\{\widetilde{\mathcal{B}}_2 = \nabla \cdot \mathbf{K} - \dfrac{\partial A}{\partial t}, \mathcal{R}_2\right\} \supset \{\mathcal{R}_2\}$;

for $D(2, 2, 1)$: $\{\mathcal{K}_1 = \dfrac{\partial \mathbf{R}}{\partial t} + \nabla \times \mathbf{K}, \widetilde{\mathcal{R}}_1, \mathcal{A}_1\} \supset (\{\widetilde{\mathcal{R}}_1\}, \{\mathcal{A}_1\})$

and $\{\widetilde{\mathcal{B}}_2, \mathcal{W}_2, \mathcal{R}_2\} \supset (\{\widetilde{\mathcal{B}}_2, \mathcal{R}_2\}, \{\mathcal{W}_2, \mathcal{R}_2\}, \{\mathcal{R}_2\})$;

for $D(3, 1, 1)$: $\left\{\mathcal{N}_2 = \dfrac{\partial \mathbf{W}}{\partial t} + \nabla \times \mathbf{N}, \mathcal{W}_2, \mathcal{R}_2, \mathcal{B}_2\right\} \supset \left\{\mathcal{W}_2, \mathcal{R}_1, \mathcal{B}_2 = \dfrac{\partial A}{\partial t}\right\}$

$\supset (\{\mathcal{B}_2, \mathcal{R}_2\}, \{\mathcal{W}_2, \mathcal{R}_2\}, \{\mathcal{R}_2\})$,

and $\left\{\mathcal{C}_1 = \dfrac{\partial B}{\partial t} - \nabla \cdot \mathbf{N}, \mathcal{U}_1, \mathcal{A}_1\right\} \supset \{\mathcal{U}_1, \mathcal{A}_1\} \supset \{\mathcal{A}_1\}$. (A.1)

Transformation properties of the forms presented in equations (A.1) are described by relations (4) where capital letters should be replaced by calligraphic ones. The forms given in brackets are closed w.r.t. the Galilei transformations.

Equating differential forms given in (A.1) to vectors with the same transformation properties or to zero, we obtain systems of linear first-order equations for Galilei vector fields. Thus, starting with representation $D(3, 1, 1)$, equating $\mathcal{N}_1$, $\mathcal{W}_1$, $\mathcal{R}_1$ and $\mathcal{B}$ to zero and $\mathcal{C}$, U$_\infty$, $\mathcal{A}$ to components of 5-current j^0, **j**, j^4, we obtain system (37).

Note that there are also tensorial differential forms, namely,

$$Y_{ab} = \nabla_a R_b + \nabla_b R_a, \qquad L_{ab} = \nabla_a N_b + \nabla_b N_a, \qquad Z^1_{ab} = \nabla_a U_b + \nabla_b U_a,$$
$$R_{ab} = \nabla_a W_b + \nabla_b W_a, \qquad Z^2_{ab} = \nabla_a K_b + \nabla_b K_a - R_{ab}, \qquad T_{ab} = \nabla_a K_b + \nabla_b K_a \tag{A.2}$$

which transform in a covariant manner under the Galilei transformations provided **R**, **U**, **W**, **K** and **N** are transformed in accordance with (4). To present invariant sets, which include (A.2), we need the forms given in (A.1) and also the following scalar and vector forms:

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$$G = \frac{\partial B}{\partial t}, \quad D = \frac{\partial C}{\partial t}, \quad \mathbf{G} = \frac{\partial \mathbf{W}}{\partial t}, \quad \mathbf{F} = \frac{\partial \mathbf{N}}{\partial t}, \quad \mathbf{P} = \frac{\partial \mathbf{R}}{\partial t}, \quad \mathbf{T} = \frac{\partial \mathbf{K}}{\partial T},$$

$$\mathbf{X} = \nabla \times \mathbf{W}, \quad \mathbf{S} = \frac{\partial \mathbf{K}}{\partial t} - \mathbf{G}, \quad \mathbf{M} = \frac{\partial \mathbf{R}}{\partial t} + \nabla B, \quad \mathbf{J} = \frac{\partial \mathbf{U}}{\partial t} + \nabla C. \tag{A.3}$$

The related sets indecomposable w.r.t. the Galilei transformations are enumerated in the following formula:

$$\{Y_{ab}\}, \quad \left\{Z_{ab}^1, \mathcal{R}_1\right\}, \quad \{R_{ab}, Y_{ab}, \mathcal{R}_2\}, \quad \left\{Z_{ab}^2, \mathcal{R}_2\right\},$$

$$\{\mathbf{M}, Y_{ab}\}, \quad \{\mathbf{P}, Y_{ab}, \mathcal{R}_2\}, \quad \{\mathbf{G}, \mathbf{M}, Y_{ab}\}, \quad \left\{D, Z_{ab}^1, \mathcal{R}_1, \mathbf{J}, \widetilde{\mathcal{B}}\right\},$$

$$\left\{\mathbf{J}, \widetilde{\mathcal{B}}_1, \mathcal{R}_1, Z_{ab}^1\right\}, \quad \{\mathbf{G}, R_{ab}, Y_{ab}, \mathcal{R}_2, \mathbf{P}, \mathcal{U}_1\}, \quad \left\{\mathbf{S}, Z_{ab}^2, \mathcal{R}_1, \mathcal{U}_2 - \mathcal{K}_2, \widetilde{\mathcal{B}}_1\right\}, \tag{A.4}$$

$$\{T_{ab}, R_{ab}, \mathbf{X}, \mathcal{R}_1\}, \quad \{T_{ab}, R_{ab}, \mathbf{X}, \mathcal{R}_1\}, \quad \{T_{ab}, R_{ab}, \mathbf{X}, \mathbf{S}, \mathcal{R}_1, \mathcal{K}_2 - \mathbf{P}, \widetilde{\mathcal{B}}\},$$

$$\{Y_{ab}, R_{ab}, L_{ab}, \mathcal{R}_2, \mathbf{P}, \mathbf{X}\}, \quad \{Y_{ab}, R_{ab}, L_{ab}, \mathcal{R}_2, \mathbf{P}, \mathbf{X}, \mathbf{F}, \mathbf{M}, \mathbf{G}, G\}.$$

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